

A note on solutions of linear systems

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Abstract

In this paper we will consider Rohde's general form of $\{1\}$ -inverse of a matrix A . The necessary and sufficient condition for consistency of a linear system $Ax = c$ will be represented. We will also be concerned with the minimal number of free parameters in Penrose's formula $x = A^{(1)}c + (I - A^{(1)}A)y$ for obtaining the general solution of the linear system. This results will be applied for finding the general solution of various homogenous and non-homogenous linear systems as well as for different types of matrix equations.

Keywords: Generalized inverses, linear systems, matrix equations

1. Introduction

In this paper we consider non-homogeneous linear system in n variables

$$Ax = c, \quad (1)$$

where A is an $m \times n$ matrix over the field \mathbb{C} of rank a and c is an $m \times 1$ matrix over \mathbb{C} . The set of all $m \times n$ matrices over the complex field \mathbb{C} will be denoted by $\mathbb{C}^{m \times n}$, $m, n \in \mathbb{N}$. The set of all $m \times n$ matrices over the complex field \mathbb{C} of rank a will be denoted by $\mathbb{C}_a^{m \times n}$. For simplicity of notation, we will write $A_{i \rightarrow} (A_{\downarrow j})$ for the i^{th} row (the j^{th} column) of the matrix $A \in \mathbb{C}^{m \times n}$.

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Any matrix X satisfying the equality $AXA = A$ is called $\{1\}$ -inverse of A and is denoted by $A^{(1)}$. The set of all $\{1\}$ -inverses of the matrix A is denoted by $A\{1\}$. It can be shown that $A\{1\}$ is not empty. If the $n \times n$ matrix A is invertible, then the equation $AXA = A$ has exactly one solution A^{-1} , so the only $\{1\}$ -inverse of the matrix A is its inverse A^{-1} , i.e. $A\{1\} = \{A^{-1}\}$. Otherwise, $\{1\}$ -inverse of the matrix A is not uniquely determined. For more informations about $\{1\}$ -inverses and various generalized inverses we recommend A.Ben-Israel and T.N.E. Greville [1] and S.L. Campbell and C.D. Meyer [2].

For each matrix $A \in \mathbb{C}_a^{m \times n}$ there are regular matrices $P \in \mathbb{C}^{n \times n}$ and $Q \in \mathbb{C}^{m \times m}$ such that

$$QAP = E_a = \left[\begin{array}{c|c} I_a & 0 \\ \hline 0 & 0 \end{array} \right], \quad (2)$$

where I_a is $a \times a$ identity matrix. It can be easily seen that every $\{1\}$ -inverse of the matrix A can be represented in the form

$$A^{(1)} = P \left[\begin{array}{c|c} I_a & U \\ \hline V & W \end{array} \right] Q \quad (3)$$

where $U = [u_{ij}]$, $V = [v_{ij}]$ and $W = [w_{ij}]$ are arbitrary matrices of corresponding dimensions $a \times (m - a)$, $(n - a) \times a$ and $(n - a) \times (m - a)$ with mutually independent entries, see C. Rohde [8] and V. Perić [7].

We will generalize the results of N. S. Urquhart [9]. Firstly, we explore the minimal numbers of free parameters in Penrose's formula

$$x = A^{(1)}c + (I - A^{(1)}A)y$$

for obtaining the general solution of the system (1). Then, we consider relations among the elements of $A^{(1)}$ to obtain the general solution in the form $x = A^{(1)}c$ of the system (1) for $c \neq 0$. This construction has previously been used by B. Malešević and B. Radičić [3] (see also [4] and [5]). At the end of this paper we will give an application of this results to the matrix equation $AXB = C$.

2. The main result

In this section we indicate how technique of an $\{1\}$ -inverse may be used to obtain the necessary and sufficient condition for an existence of a general solution of a non-homogeneous linear system.

Lemma 2.1. *The non-homogeneous linear system (1) has a solution if and only if the last $m - a$ coordinates of the vector $c' = Qc$ are zeros, where $Q \in \mathbb{C}^{m \times m}$ is regular matrix such that (2) holds.*

Proof: The proof follows immediately from Kroneker–Capelli theorem. We provide a new proof of the lemma by using the $\{1\}$ -inverse of the system matrix A . The system (1) has a solution if and only if $c = AA^{(1)}c$, see R. Penrose [6]. Since $A^{(1)}$ is described by the equations (3), it follows that

$$AA^{(1)} = AP \left[\begin{array}{c|c} I_a & U \\ \hline V & W \end{array} \right] Q = Q^{-1} \left[\begin{array}{c|c} I_a & U \\ \hline 0 & 0 \end{array} \right] Q.$$

Hence, we have the following equivalences

$$\begin{aligned} c = AA^{(1)}c &\iff (I - AA^{(1)})c = 0 \iff \left(Q^{-1}Q - Q^{-1} \left[\begin{array}{c|c} I_a & U \\ \hline 0 & 0 \end{array} \right] Q \right) c = 0 \\ &\iff Q^{-1} \left[\begin{array}{c|c} 0 & -U \\ \hline 0 & I_{n-a} \end{array} \right] \underbrace{Qc}_{c'} = 0 \iff \left[\begin{array}{c|c} 0 & -U \\ \hline 0 & I_{n-a} \end{array} \right] c' = 0 \\ c' = \begin{bmatrix} c'_a \\ c'_{n-a} \end{bmatrix} &\iff \left[\begin{array}{c|c} 0 & -U \\ \hline 0 & I_{n-a} \end{array} \right] \begin{bmatrix} c'_a \\ c'_{n-a} \end{bmatrix} = 0 \iff \begin{bmatrix} -Uc'_{n-a} \\ c'_{n-a} \end{bmatrix} = 0 \\ &\iff c'_{n-a} = 0. \end{aligned}$$

Furthermore, we conclude $c = AA^{(1)}c \iff c'_{n-a} = 0$. \square

Theorem 2.2. *The vector*

$$x = A^{(1)}c + (I - A^{(1)}A)y,$$

$y \in \mathbb{C}^{n \times 1}$ is an arbitrary column, is the general solution of the system (1), if and only if the $\{1\}$ -inverse $A^{(1)}$ of the system matrix A has the form (3) for arbitrary matrices U and W and the rows of the matrix $V(c'_a - y'_a) + y'_{(n-a)}$ are free parameters, where $Qc = c' = \begin{bmatrix} c'_a \\ 0 \end{bmatrix}$ and $P^{-1}y = y' = \begin{bmatrix} y'_a \\ y'_{n-a} \end{bmatrix}$.

Proof: Since $\{1\}$ -inverse $A^{(1)}$ of the matrix A has the form (3), the solution of the system $x = A^{(1)}c + (I - A^{(1)}A)y$ can be represented in the form

$$\begin{aligned} x &= P \left[\begin{array}{c|c} I_a & U \\ \hline V & W \end{array} \right] Qc + \left(I - P \left[\begin{array}{c|c} I_a & U \\ \hline V & W \end{array} \right] QA \right) y \\ &= P \left[\begin{array}{c|c} I_a & U \\ \hline V & W \end{array} \right] c' + \left(I - P \left[\begin{array}{c|c} I_a & U \\ \hline V & W \end{array} \right] QAPP^{-1} \right) y. \end{aligned}$$

According to Lemma 2.1 and from (2) we have

$$x = P \left[\begin{array}{c|c} I_a & U \\ \hline V & W \end{array} \right] \left[\begin{array}{c} c'_a \\ 0 \end{array} \right] + \left(I - P \left[\begin{array}{c|c} I_a & U \\ \hline V & W \end{array} \right] \left[\begin{array}{c|c} I_a & 0 \\ \hline 0 & 0 \end{array} \right] P^{-1} \right) y.$$

Furthermore, we obtain

$$\begin{aligned} x &= P \left[\begin{array}{c} c'_a \\ V c'_a \end{array} \right] + \left(I - P \left[\begin{array}{c|c} I_a & 0 \\ \hline V & 0 \end{array} \right] P^{-1} \right) \left[\begin{array}{c} y_a \\ y_{n-a} \end{array} \right] \\ &= P \left[\begin{array}{c} c'_a \\ V c'_a \end{array} \right] + \left(P P^{-1} - P \left[\begin{array}{c|c} I_a & 0 \\ \hline V & 0 \end{array} \right] P^{-1} \right) \left[\begin{array}{c} y_a \\ y_{n-a} \end{array} \right] \\ &= P \left[\begin{array}{c} c'_a \\ V c'_a \end{array} \right] + P \left(I - \left[\begin{array}{c|c} I_a & 0 \\ \hline V & 0 \end{array} \right] \right) P^{-1} \left[\begin{array}{c} y_a \\ y_{n-a} \end{array} \right] \\ &= P \left[\begin{array}{c} c'_a \\ V c'_a \end{array} \right] + P \left[\begin{array}{c|c} 0 & 0 \\ \hline -V & I_{n-a} \end{array} \right] \left[\begin{array}{c} y'_a \\ y'_{n-a} \end{array} \right], \end{aligned}$$

where $y' = P^{-1}y$. We now conclude

$$x = P \left(\left[\begin{array}{c} c'_a \\ V c'_a \end{array} \right] + \left[\begin{array}{c} 0 \\ -V y'_a + y'_{n-a} \end{array} \right] \right) = P \left[\begin{array}{c} c'_a \\ V(c'_a - y'_a) + y'_{n-a} \end{array} \right].$$

Therefore, since matrix P is regular we deduce that $P \left[\begin{array}{c} c'_a \\ V(c'_a - y'_a) + y'_{n-a} \end{array} \right]$ is the general solution of the system (1) if and only if the rows of the matrix $V(c'_a - y'_a) + y'_{n-a}$ are $n - a$ free parameters. \square

Corollary 2.3. *The vector*

$$x = (I - A^{(1)}A)y,$$

$y \in \mathbb{C}^{n \times 1}$ is an arbitrary column, is the general solution of the homogeneous linear system $Ax = 0$, $A \in \mathbb{C}^{m \times n}$, if and only if the $\{1\}$ -inverse $A^{(1)}$ of the system matrix A has the form (3) for arbitrary matrices U and W and the rows of the matrix $-V y'_a + y'_{(n-a)}$ are free parameters, where $P^{-1}y = y' = \left[\begin{array}{c} y'_a \\ y'_{n-a} \end{array} \right]$.

Example 2.4. *Consider the homogeneous linear system*

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ 4x_1 + 5x_2 + 6x_3 &= 0. \end{aligned} \tag{4}$$

The system matrix is

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}.$$

For regular matrices

$$Q = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \text{ and } P = \begin{bmatrix} 1 & \frac{2}{3} & 1 \\ 0 & -\frac{1}{3} & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

equality (2) holds. Rohde's general $\{1\}$ -inverse $A^{(1)}$ of the system matrix A is of the form

$$A^{(1)} = P \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ v_{11} & v_{12} \end{bmatrix} Q$$

According to Corollary 2.3 the general solution of the system (4) is of the form

$$x = P \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline -v_{11} & -v_{12} & 1 \end{array} \right] P^{-1} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix},$$

where

$$P^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore, we obtain

$$\begin{aligned} x &= P \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline -v_{11} & -v_{12} & 1 \end{array} \right] \begin{bmatrix} y_1 + 2y_2 + 3y_3 \\ -3y_2 - 6y_3 \\ y_3 \end{bmatrix} \\ &= P \begin{bmatrix} 0 \\ 0 \\ -v_{11}y_1 - (2v_{11} - 3v_{12}y_2 - (3v_{11} - 6v_{12} - 1)y_3) \end{bmatrix}. \end{aligned}$$

If we take $\alpha = -v_{11}y_1 - (2v_{11} - 3v_{12}y_2 - (3v_{11} - 6v_{12} - 1)y_3)$ as a parameter we get the general solution

$$x = \begin{bmatrix} 1 & \frac{2}{3} & 1 \\ 0 & -\frac{1}{3} & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \alpha \end{bmatrix} = \begin{bmatrix} \alpha \\ -2\alpha \\ \alpha \end{bmatrix}.$$

Corollary 2.5. *The vector*

$$x = A^{(1)}c$$

is the general solution of the system (1), if and only if the $\{1\}$ -inverse $A^{(1)}$ of the system matrix A has the form (3) for arbitrary matrices U and W and the rows of the matrix Vc'_a are free parameters, where $Qc = c' = \begin{bmatrix} c'_a \\ 0 \end{bmatrix}$.

Remark 2.6. *Similar result can be found in paper B. Malešević and B. Radičić [3].*

Example 2.7. *Consider the non-homogeneous linear system*

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 7 \\ 4x_1 + 5x_2 + 6x_3 &= 8. \end{aligned} \tag{5}$$

According to Corollary 2.5 the general solution of the system (5) is of the form

$$x = P \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ v_{11} & v_{12} \end{bmatrix} Q \begin{bmatrix} 7 \\ 8 \end{bmatrix} = P \begin{bmatrix} 7 \\ -20 \\ 7v_{11} - 20v_{12} \end{bmatrix}.$$

If we take $\alpha = 7v_{11} - 20v_{12}$ as a parameter we obtain the general solution of the system

$$x = P \begin{bmatrix} 7 \\ -20 \\ \alpha \end{bmatrix} = \begin{bmatrix} 1 & \frac{2}{3} & 1 \\ 0 & -\frac{1}{3} & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ -20 \\ \alpha \end{bmatrix} = \begin{bmatrix} -\frac{19}{3} + \alpha \\ \frac{20}{3} - 2\alpha \\ \alpha \end{bmatrix}.$$

We are now concerned with the matrix equation

$$AX = C, \tag{6}$$

where $A \in \mathbb{C}^{m \times n}$, $X \in \mathbb{C}^{n \times k}$ and $C \in \mathbb{C}^{m \times k}$.

Lemma 2.8. *The matrix equation (6) has a solution if and only if the last $m - a$ rows of the matrix $C' = QC$ are zeros, where $Q \in \mathbb{C}^{m \times m}$ is regular matrix such that (2) holds.*

Proof: If we write $X = [X_{\downarrow 1} \ X_{\downarrow 2} \ \dots \ X_{\downarrow k}]$ and $C = [C_{\downarrow 1} \ C_{\downarrow 2} \ \dots \ C_{\downarrow k}]$, then we can observe the matrix equation (6) as the system of matrix equations

$$AX_{\downarrow 1} = C_{\downarrow 1}$$

$$AX_{\downarrow 2} = C_{\downarrow 2}$$

$$\vdots$$

$$AX_{\downarrow k} = C_{\downarrow k}.$$

Each of the matrix equation $AX_{\downarrow i} = C_{\downarrow i}$, $1 \leq i \leq k$, by Lemma 2.1 has solution if and only if the last $m - a$ coordinates of the vector $C'_{\downarrow i} = QC_{\downarrow i}$ are zeros. Thus, the previous system has solution if and only if the last $m - a$ rows of the matrix $C' = QC$ are zeros, which establishes that the matrix equation (6) has solution if and only if all entries of the last $m - a$ rows of the matrix C' are zeros. \square

Theorem 2.9. *The matrix*

$$X = A^{(1)}C + (I - A^{(1)}A)Y \in \mathbb{C}^{n \times k},$$

$Y \in \mathbb{C}^{n \times k}$ is an arbitrary matrix, is the general solution of the matrix equation (6) if and only if the $\{1\}$ -inverse $A^{(1)}$ of the system matrix A has the form (3) for arbitrary matrices U and W and the entries of the matrix

$$V(C'_a - Y'_a) + Y'_{(n-a)}$$

are mutually independent free parameters, where $QC = C' = \begin{bmatrix} C'_a \\ 0 \end{bmatrix}$ and

$$P^{-1}Y = Y' = \begin{bmatrix} Y'_a \\ Y'_{n-a} \end{bmatrix}.$$

Proof: Applying the Theorem 2.2 on the each system $AX_{\downarrow i} = C_{\downarrow i}$, $1 \leq i \leq k$, we obtain that

$$X_{\downarrow i} = P \left[\frac{C'_{a\downarrow i}}{V(C'_{a\downarrow i} - Y'_{a\downarrow i}) + Y'_{n-a\downarrow i}} \right]$$

is the general solution of the system if and only if the rows of the matrix $V(C'_{a\downarrow i} - Y'_{a\downarrow i}) + Y'_{n-a\downarrow i}$ are $n - a$ free parameters. Assembling these individual solutions together we get that

$$X = P \left[\frac{C'_a}{V(C'_a - Y'_a) + Y'_{n-a}} \right]$$

is the general solution of the matrix equation (6) if and only if entries of the matrix $V(C'_a - Y'_a) + Y'_{n-a}$ are $(n-a)k$ mutually independent free parameters. \square

From now on we proceed with the study of the non-homogeneous linear system of the form

$$xB = d, \quad (7)$$

where B is an $n \times m$ matrix over the field \mathbb{C} of rank b and d is an $1 \times m$ matrix over \mathbb{C} . Let $R \in \mathbb{C}^{n \times n}$ and $S \in \mathbb{C}^{m \times m}$ be regular matrices such that

$$RBS = E_b = \left[\begin{array}{c|c} I_b & 0 \\ \hline 0 & 0 \end{array} \right]. \quad (8)$$

An $\{1\}$ -inverse of the matrix B can be represented in the Rohde's form

$$B^{(1)} = S \left[\begin{array}{c|c} I_b & M \\ \hline N & K \end{array} \right] R \quad (9)$$

where $M = [u_{ij}]$, $N = [v_{ij}]$ and $K = [w_{ij}]$ are arbitrary matrices of corresponding dimensions $b \times (n - b)$, $(m - b) \times b$ and $(m - b) \times (n - b)$ with mutually independent entries.

Lemma 2.10. *The non-homogeneous linear system (7) has a solution if and only if the last $m - b$ elements of the row $d' = dS$ are zeros, where $S \in \mathbb{C}^{m \times m}$ is regular matrix such that (8) holds.*

Proof: By transposing the system (7) we obtain system $B^T x^T = d^T$ and by transposing the matrix equation (8) we obtain that $S^T B^T R^T = E_b$. According to Lemma 2.1 the system $B^T x^T = d^T$ has solution if and only if the last $m - b$ coordinates of the vector $S^T d^T$ are zeros, i.e. if and only if the last $m - b$ elements of the row $d' = dS$ are zeros. \square

Theorem 2.11. *The row*

$$x = dB^{(1)} + y(I - BB^{(1)}),$$

$y \in \mathbb{C}^{1 \times n}$ is an arbitrary row, is the general solution of the system (7), if and only if the $\{1\}$ -inverse $B^{(1)}$ of the system matrix B has the form (9) for arbitrary matrices N and K and the columns of the matrix $(d'_b - y'_b)M + y'_{n-b}$ are free parameters, where $dS = d' = [d'_b \mid 0]$ and $yR^{-1} = y' = [y'_b \mid y'_{n-b}]$.

Proof: The basic idea of the proof is to transpose the system (7) and to apply the Theorem 2.2. The $\{1\}$ -inverse of the matrix B^T is equal to a transpose of the $\{1\}$ -inverse of the matrix B . Hence, we have

$$(B^T)^{(1)} = (B^{(1)})^T = \left(S \left[\begin{array}{c|c} I_b & M \\ \hline N & K \end{array} \right] R \right)^T = R^T \left[\begin{array}{c|c} I_b & N^T \\ \hline M^T & K^T \end{array} \right] S^T.$$

We can now proceed analogously to the proof of the Theorem 2.2 to obtain that

$$x^T = R^T \left[\frac{d_b'^T}{M^T(d_b'^T - y_b'^T) + y_{n-b}'^T} \right]$$

is the general solution of the system $B^T x^T = d^T$ if and only if the rows of the matrix $M^T(d_b'^T - y_b'^T) + y_{n-b}'^T$ are $n - b$ free parameters. Therefore,

$$x = [d_b' \mid (d_b' - y_b')M + y_{n-b}'] R$$

is the general solution of the system (7) if and only if the columns of the matrix $(d_b' - y_b')M + y_{n-b}'$ are $n - b$ free parameters. \square

Analogous corollaries hold for the Theorem 2.11.

We now deal with the matrix equation

$$XB = D, \tag{10}$$

where $X \in \mathbb{C}^{k \times n}$, $B \in \mathbb{C}^{n \times m}$ and $D \in \mathbb{C}^{k \times m}$.

Lemma 2.12. *The matrix equation (10) has a solution if and only if the last $m - b$ columns of the matrix $D' = DS$ are zeros, where $S \in \mathbb{C}^{m \times m}$ is regular matrix such that (8) holds.*

Theorem 2.13. *The matrix*

$$X = DB^{(1)} + Y(I - BB^{(1)}) \in \mathbb{C}^{k \times n},$$

$Y \in \mathbb{C}^{k \times n}$ is an arbitrary matrix, is the general solution of the matrix equation (10) if and only if the $\{1\}$ -inverse $B^{(1)}$ of the system matrix B has the form (9) for arbitrary matrices N and K and the entries of the matrix

$$V(D_b' - Y_b')M + Y_{(n-b)}'$$

are mutually independent free parameters, where $DS = D' = [D_b' \mid 0]$ and $YR^{-1} = Y' = [Y_b' \mid Y_{n-b}']$.

3. An application

In this section we will briefly sketch properties of the general solution of the matrix equation

$$AXB = C, \quad (11)$$

where $A \in \mathbb{C}^{m \times n}$, $X \in \mathbb{C}^{n \times k}$, $B \in \mathbb{C}^{k \times l}$ and $C \in \mathbb{C}^{m \times l}$. If we denote by Y matrix product XB , then the matrix equation (11) becomes

$$AY = C. \quad (12)$$

According to the Theorem 2.9 the general solution of the system (12) can be presented as a product of the matrix P and the matrix which has the first $a = \text{rank}(A)$ rows same as the matrix QC and the elements of the last $m - a$ rows are $(m - a)n$ mutually independent free parameters, P and Q are regular matrices such that $PAQ = E_a$. Thus, we are now turning on to the system of the form

$$XB = D. \quad (13)$$

By the Theorem 2.13 we conclude that the general solution of the system (13) can be presented as a product of the matrix which has the first $b = \text{rank}(B)$ columns equal to the first b columns of the matrix DS and the rest of the columns have mutually independent free parameters as entries, and the matrix R , for regular matrices R and S such that $RBS = E_b$. Therefore, the general solution of the system (11) is of the form

$$X = P \left[\begin{array}{c|c} G_{ab} & F \\ \hline H & L \end{array} \right] R,$$

where G_{ab} is a submatrix of the matrix QCS corresponding to the first a rows and the first b columns and the entries of the matrices F , H and L are $nk - ab$ free parameters. We will illustrate this on the following example.

Example 3.1. *We consider the matrix equation*

$$AXB = C,$$

where $A = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 2 & 1 \\ -2 & -4 & -2 \end{bmatrix}$. If we take $Y = XB$, we obtain the system

$$AY = C.$$

It is easy to check that the matrix A is of the rank $a = 1$ and for matrices $Q = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ the equality $QAP = E_a$ holds. Based on the Theorem 2.9, the equation $AY = C$ can be rewritten in the system form

$$\begin{aligned} AY_{\downarrow 1} &= \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\ AY_{\downarrow 2} &= \begin{bmatrix} 2 \\ -4 \end{bmatrix} \\ AY_{\downarrow 3} &= \begin{bmatrix} 1 \\ -2 \end{bmatrix}. \end{aligned}$$

Combining the Theorem 2.2 with the equality

$$\begin{bmatrix} c'_1 & c'_2 & c'_3 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ -2 & -4 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

yields

$$\begin{aligned} Y_{\downarrow 1} &= P \begin{bmatrix} 1 \\ \underbrace{v - vz_{11} + z_{21}}_{\alpha} \end{bmatrix} \\ Y_{\downarrow 2} &= P \begin{bmatrix} 2 \\ \underbrace{2v - 2vz_{12} + z_{22}}_{\beta} \end{bmatrix} \\ Y_{\downarrow 3} &= P \begin{bmatrix} 1 \\ \underbrace{v - vz_{13} + z_{23}}_{\gamma} \end{bmatrix}, \end{aligned}$$

for an arbitrary matrix $Z = \begin{bmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & -z_{22} & z_{23} \end{bmatrix}$. Therefore, the general solution of the system $AY = C$ is

$$Y = P \begin{bmatrix} 1 & 2 & 1 \\ \alpha & \beta & \gamma \end{bmatrix}.$$

From now on, we consider the system

$$XB = D$$

for

$$D = P \begin{bmatrix} 1 & 2 & 1 \\ \alpha & \beta & \gamma \end{bmatrix} = \begin{bmatrix} 1 + 2\alpha & 2 + 2\beta & 1 + 2\gamma \\ \alpha & \beta & \gamma \end{bmatrix}.$$

There are regular matrices $R = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ and $S = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ such that $RBS = E_b$ holds. Since the rank of the matrix B is $b = 1$, according to the Lemma 2.12 all entries of the last two columns of the matrix $D' = DS$ are zeros, i.e. we have $\gamma = \alpha$, $\beta = 2\alpha$. Hence, we get that the matrix D' is of the form $D' = \begin{bmatrix} 1+2\alpha & 0 & 0 \\ \alpha & 0 & 0 \end{bmatrix}$. Applying the Theorem 2.13, we obtain

$$X = \begin{bmatrix} 1+2\alpha & \underbrace{(1+2\alpha - t_{11})m_{11} + t_{12}}_{\beta_1} & \underbrace{(1+2\alpha - t_{11})m_{12} + t_{13}}_{\beta_2} \\ \alpha & \underbrace{(\alpha - t_{21})m_{11} + t_{22}}_{\gamma_1} & \underbrace{(\alpha - t_{12})m_{12} + t_{23}}_{\gamma_2} \end{bmatrix} R,$$

for an arbitrary matrix $T = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \end{bmatrix}$. Finally, the solution of the system $AXB = C$ is

$$X = \begin{bmatrix} 1+2\alpha - \beta_1 - \beta_2 & \beta_1 & \beta_2 \\ \alpha - \gamma_1 - \gamma_2 & \gamma_1 & \gamma_2 \end{bmatrix}.$$

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